Lecture 4: Channel Properties

I. Channel Fidelities

Assumption: Input space = output space

Goal: measure, how good the channel transmits the inputs, i.e. compare the output of the channel to the input.

4.1 Channel fidelity for pure state inputs

Let Q be a quantum channel on $\mathcal{H}$ and let $|\psi\rangle \in \mathcal{H}$ be pure state input.

Then we define

$$F_p(|\psi\rangle, Q) := \langle\psi| Q(|\psi\rangle\langle\psi|) |\psi\rangle$$

(overlap of the output of the channel with the input, i.e. probability to measure $|\psi\rangle$ at the output)

The pure state channel fidelity is given by

$$F_p(Q) := \min_{|\psi\rangle \in \mathcal{H}} F_p(|\psi\rangle, Q).$$

For a channel with error-operators $\{A_i : i \in I_Q\}$ we get

$$F_p(|\psi\rangle, Q) = \langle\psi| \, Q(|\psi\rangle\langle\psi|) \, |\psi\rangle$$

$$= \langle\psi| \Big(\sum_{i \in I_Q} A_i |\psi\rangle\langle\psi| A_i^\dagger \Big) |\psi\rangle$$

$$= \sum_{i \in I_Q} |\langle\psi| A_i |\psi\rangle|^2$$

4.2 Distance of density matrices
(Uhlmann 76, Josza 94)

$$F_m(\rho_1, \rho_2) := \Big(\mathrm{Tr} \sqrt{\sqrt{\rho_1} \cdot \rho_2 \cdot \sqrt{\rho_1}} \Big)^2$$

$$= \max_{|\psi_1\rangle, |\psi_2\rangle} |\langle\psi_1|\psi_2\rangle|^2$$

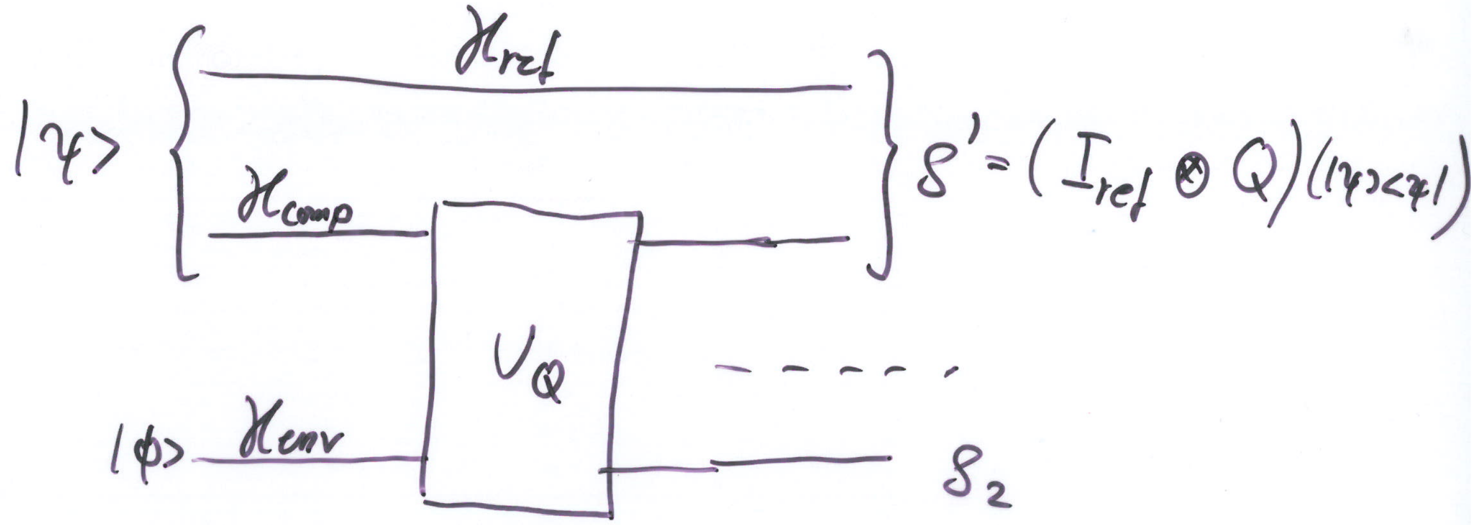
where $|\psi_i\rangle$ is any purification of ρ_i.

For indg. $\rho_1 = |\phi_1\rangle\langle\phi_1|$, we have

$$F_m(|\phi_1\rangle\langle\phi_1|, \rho_2) = \langle\phi_1|\rho_2|\phi_1\rangle.$$

$\Rightarrow$ for pure state inputs, $F_p(|\psi\rangle, Q) = F_m(|\psi\rangle, Q)$

This fidelity does not apply to the situation when the channel is operating on only a part of our system, but we want to look at the whole system.



4.3 Entanglement fidelity

$$F_e(|\psi\rangle, Q) := F_p(|\psi\rangle, I_{ref} \otimes Q)$$

$$= \langle\psi| (I_{ref} \otimes Q)(|\psi\rangle\langle\psi|) |\psi\rangle$$

$$F_e(Q) = \min_{|\psi\rangle \in \mathcal{H}_{ref} \otimes \mathcal{H}_{comp}} F_e(|\psi\rangle, Q)$$

(4)

Schumacher showed that $F_e(Q)$ is independent of the reference system and can be computed as follows:

Let $|\psi\rangle \in \mathcal{H}_{ref} \otimes \mathcal{H}_{comp}$ and be Q a quantum channel on $\mathcal{H}_{comp}$ with error-operators $\{A_i : i \in I_Q\}$.

Then $$F_e(|\psi\rangle, Q) = \sum_{i \in I_Q} |\mathrm{Tr}(\rho_{comp} \cdot A_i)|^2$$

where $$\rho_{comp} = \mathrm{Tr}_{ref}(|\psi\rangle\langle\psi|)$$

$$\Rightarrow \quad F_e(Q) := \min_{\rho_{comp} \in \mathcal{H}_{comp}} \sum_{i \in I_Q} |\mathrm{Tr}(\rho_{comp} \cdot A_i)|^2$$

$\wedge\!\!|$

$$F_p(Q) := \min_{|\psi\rangle \in \mathcal{H}_{comp}} \sum_{i \in I_Q} |\mathrm{Tr}(|\psi\rangle\langle\psi| \cdot A_i)|^2$$

Schumacher: $$F_e(\rho_{comp}, Q) \le F_m(\rho_{comp}, Q(\rho_{comp}))$$

Knill & Laflamme showed the following relation:

$$F_e(Q) \geq \left(3F_p(Q)-1\right)/2$$

$\Rightarrow$ $F_p(Q) > 1/3$, then $F_e(Q) > 0$

$F_p(Q) = 1$, then $F_e(Q) = 1$

II: Channel capacity

rate of an encoding scheme:

Let $C: \mathcal{H}_{in} \to \mathcal{H}_{out}$ be an encoding operation. Then the rate of C is defined as

$$R_C = \frac{\operatorname{ld}(\dim \mathcal{H}_{in})}{\operatorname{ld}(\dim \mathcal{H}_{out})}$$

In particular for $\mathcal{H}_{in} = (\mathbb{C}^2)^{\otimes k}$ k qubits

$\mathcal{H}_{out} = (\mathbb{C}^2)^{\otimes n}$ n qubits

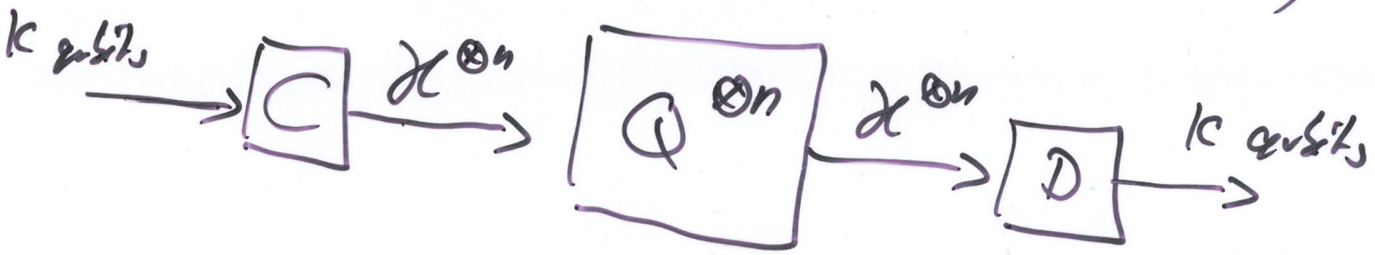
$$\Rightarrow R = \frac{k}{n}$$

Quantum Channel Capacity

Let Q be a channel on $\mathcal{H}$,

C be an encoding operation $(\mathbb{C}^2)^{\otimes k} \to \mathcal{H}^{\otimes n}$

D be a decoding operation $\mathcal{H}^{\otimes n} \to (\mathbb{C}^2)^{\otimes k}$



$$\mathcal{Q}(Q) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \left\{ \frac{k}{n} \;\middle|\; \exists k, C, D : F_p\left(D Q^{\otimes n} C\right) > 1 - \varepsilon \right\}$$

Does not change if F_p is replaced by F_e.