Lecture 7: General Error-Correction Algorithm

(Classical Error-Correcting Codes → later)

7.1 Recall QECC conditions

(i) $\langle c_i | A_k^\dagger A_\ell | c_j \rangle = 0$ (for $i \neq j$)

(ii) $\langle c_i | A_k^\dagger A_\ell | c_i \rangle = \langle c_j | A_k^\dagger A_\ell | c_j \rangle = \alpha_{k\ell}$

equivalently

$$P_{\mathcal{C}} A_k^\dagger A_\ell P_{\mathcal{C}} = \alpha_{k\ell} \cdot P_{\mathcal{C}}$$

after Gram-Schmidt orthonormalization

	$\mathcal{C}_1 = \tilde{A}_1 \mathcal{C}$	$\mathcal{C}_k = \tilde{A}_k \mathcal{C}$
V_0	$\tilde{A}_1 \vert c_0\rangle$	$\hat{A}_k \vert c_0\rangle$
V_1	$\tilde{A}_1 \vert c_1\rangle$	$\hat{A}_k \vert c_1\rangle$
$\vdots$	$\vdots$	
V_i	$\tilde{A}_1 \vert c_i\rangle$	$\hat{A}_k \vert c_i\rangle$

the entries are mutually orthogonal

7.2 General QECC procedure

0. Check the conditions.
1. Compute an ONB of the space V_0 spanned by $\{A_k|c_0\rangle : k \in I_Q\}$
2. $\Rightarrow$ transformation of the error operators with the coefficients λ_{jk} obtained by orthonormalization

$$\tilde{A}_j = \sum_k \lambda_{jk} A_k$$

3. Find unitary transformations $\tilde{T}_j$ with

$$\forall i \in I_e : \tilde{T}_j \hat{A}_j |c_i\rangle = |c_i\rangle$$

4. Find a measurement that has eigenspaces

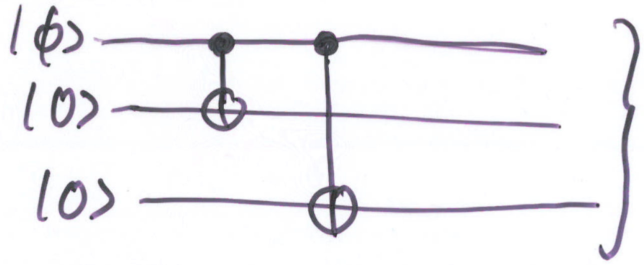
$$\tilde{\mathcal{C}}_j = \tilde{A}_j \mathcal{C} \quad \text{(plus some additional eigenspace if } \bigoplus_j \mathcal{C}_j \subsetneq \mathcal{H} \text{)}$$

Measurement projects onto one of these spaces and yields its index j.

5. Correct the error using the transformation $\tilde{T}_j$.

7.3 Example: The 3-qubit code

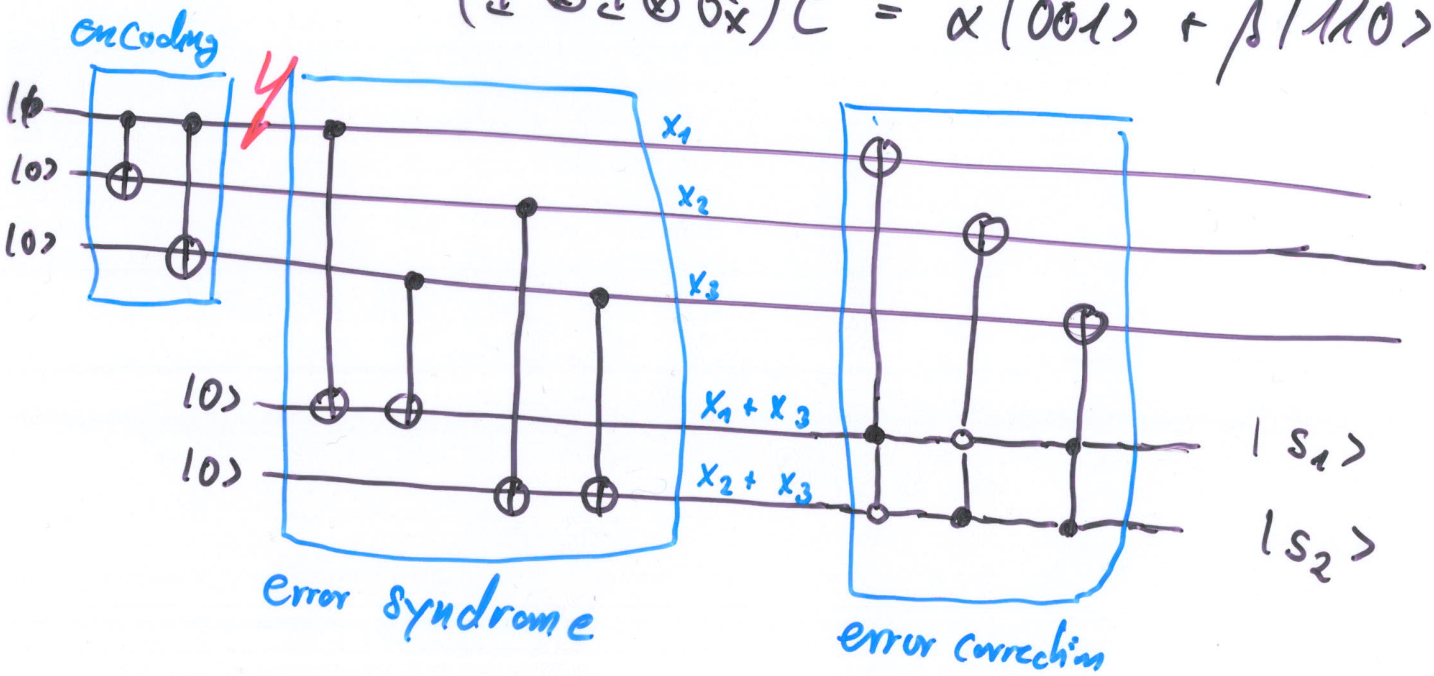
encoding: $\alpha|0\rangle + \beta|1\rangle \mapsto \alpha|000\rangle + \beta|111\rangle$



$\alpha|000\rangle + \beta|111\rangle$

error spaces:

$$(I \otimes I \otimes I)\,\mathcal{C} = \alpha|000\rangle + \beta|111\rangle$$
$$(\sigma_x \otimes I \otimes I)\,\mathcal{C} = \alpha|100\rangle + \beta|011\rangle$$
$$(I \otimes \sigma_x \otimes I)\,\mathcal{C} = \alpha|010\rangle + \beta|101\rangle$$
$$(I \otimes I \otimes \sigma_x)\,\mathcal{C} = \alpha|001\rangle + \beta|110\rangle$$



assume the error was $\mu\sigma_x \otimes I \otimes I + \lambda I \otimes \sigma_x \otimes I$

$\Rightarrow$ syndrome qubit is $\mu|10\rangle + \lambda|01\rangle$

it factors after the error correction step

$\alpha|000\rangle + \beta|111\rangle$

after error $E = \mu(X \otimes I \otimes I) + \lambda(I \otimes X \otimes I)$

$$\mu(\alpha|100\rangle + \beta|011\rangle) + \lambda(\alpha|010\rangle + \beta|101\rangle)$$

Computing the syndrome:

$$\mu(\alpha|100\rangle + \beta|011\rangle)|10\rangle$$
$$+\lambda(\alpha|010\rangle + \beta|101\rangle)|01\rangle$$

after error correction:

$$(\alpha|000\rangle + \beta|111\rangle)\otimes(\mu|10\rangle + \lambda|01\rangle)$$

the syndrome contains information about the error E, but not coefficients α, β

2.4 QECC conditions by Ekert & Macchiavello 96:

sufficient condition for QECC:

$$\forall k,l \in I_Q \ \forall i,j \in I_{\mathcal{C}}: \ \langle c_i | A_k^\dagger A_l | c_j \rangle = \delta_{ij}\delta_{kl}$$

$\Rightarrow$ all vectors $A_k|c_j\rangle$ form a subset of an ONB

2.5 non-degenerate QECC

A QECC $\mathcal{C}$ for a channel Q is non-degenerate if the matrix (α_{kl}) has full rank, or equivalently if there are error operator $\tilde{A}_k$ such that

$$\langle c_i | \tilde{A}_k^\dagger \tilde{A}_l | c_j \rangle = \delta_{ij}\delta_{kl}.$$

A non-degenerate code allows to distinguish between the different error operators as the spaces $\tilde{A}_k \mathcal{C}$ are mutually orthogonal (and none of them is zero)

7.6 Weight of an error

If an error operator is (up to permutation of the subsystems) of the form

$$A = c \cdot \mathbb{1}^{(1)} \otimes \ldots \otimes \mathbb{1}^{(n-w)} \otimes A'$$

where $c \neq 0$ and $n-w$ is maximal, then the error A has $\mathrm{wgt}(A) = w$.
Define the weight of the operator $0 \cdot \mathbb{1}$ to be zero.

7.7 t-error correcting code

A code $\mathcal{C} \leq \mathcal{H}^{\otimes n}$ is t-error correcting if the QECC conditions are fulfilled for all error operators up to weight t.

Bases for error operators

It is sufficient that the QECC conditions hold for a basis of all operators up to weight t.

Basis 1: $P_{00} = |0\rangle\langle 0|$ $\quad P_{11} = |1\rangle\langle 1|$

$P_{01} = |0\rangle\langle 1|$ $\quad P_{10} = |1\rangle\langle 0|$

better: use a basis that contains identity

7.8 Pauli errors

The Pauli matrices

$$\sigma_x = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = Y$ (note: sometimes $Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$)

together with identity $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ form a basis of all 2×2 matrices.

An n-qubit Pauli error is the tensor product of n Pauli matrices or identity. Its weight is the number of factors different from identity.

Corollary: A code is t-error correcting if it can correct all Pauli errors up to weight t.

Note that the Pauli errors up to weight t form a basis of the operators up to weight t.

7.9 Pure QECC

A QECC is pure if

$$\langle c_i | E_k^\dagger E_\ell | c_j \rangle = \delta_{ij}\, \delta_{k\ell} \qquad (*)$$

for all Pauli error E_k up to weight t.

A pure code is non-degenerate.

7.10 Quantum Hamming Bound

A pure QECC on n qubits which can correct up to t errors and dimension K fulfills the conditions

$$\underbrace{K}_{\dim \mathcal{C}} \cdot \sum_{w=0}^{t} \underbrace{3^w \binom{n}{w}}_{\substack{\text{number of}\\ n\text{-qubit Pauli}\\ \text{errors of weight } w}} \le \underbrace{2^n}_{\substack{\text{dimension of the full}\\ \text{space}}}.$$

example: encoding one qubit: $K=2$
correcting one error: $t=1$

$\Rightarrow n \geq 5 \qquad K \cdot (1 + 3 \cdot n) \le 2^n$

note: No code violating this bound is known (to me).

7.11 Detecting errors / Correcting erasures

A code $\mathcal{C} \leq \mathcal{H}^{\otimes n}$ is t-erasure correcting, if the conditions

(i) $\forall i \neq j \in I_{\mathcal{C}}: \quad \langle c_i | A | c_j \rangle = 0$

(ii) $\forall i,j \in I_{\mathcal{C}}: \quad \langle c_i | A | c_i \rangle = \langle c_j | A | c_j \rangle = \beta_A$

hold for all error operators A up to weight t.

A code that is t-error correcting can correct up to $\min\{2t, n\}$ erasures.

Proof (main idea)

An erasure on subsystem j is indicated by a state $|\varepsilon\rangle_j$ that is orthogonal to all other states.

If the positions the operators A_k and A_ℓ operate non-trivially on are different, then $A_k^\dagger A_\ell = 0$.

If the positions are the same, we can replace $A_k^\dagger A_\ell$ by A.

Note: Fixing the code $\mathcal{C}$, we get linear conditions for the operators A which correspond to erasures: and β_A

$$\langle c_i | A | c_j \rangle = 0$$

$$\langle c_i | A | c_i \rangle = \langle c_j | A | c_j \rangle = \beta_A$$

Sometimes one finds that the code $\mathcal{C}$ can "detect" the error A, but this is not necessarily true, since different operators A need not yield orthogonal spaces.

Erasure correction procedure:

- Fix the position of the error (since we get them as side information from the channel)
- Apply "standard" general error correction for all errors affecting only these positions

$\Longrightarrow$ different syndromes / error correction step for each set ~~err~~ of error positions

Note (open problem to me)

Given a code $\mathcal{C}$, we can compute the linear space of erasure operators. $\mathcal{E}(\mathcal{C})$

$$\langle c_i | A | c_j \rangle = \delta_{ij} \beta_A$$

From this space, find a set of error operators, i.e.

$$\{ A_k : \text{s.t. } A_k^+ A_\ell \in \mathcal{E}(\mathcal{C}) \text{ for all } k, \ell \}$$

The space of operator A_k is not uniquely defined by $\mathcal{E}(\mathcal{C})$, not even its dimension.