15. Lecture: Stabilizer Codes

Code $\mathcal{C} = [[n,k,d]]$ is a subspace of dimension 2^k of $(\mathbb{C}^2)^{\otimes n}$, defined as the joint +1 eigenspace of some commuting tensor products of Pauli matrices, the stabilizer group S.

Pauli operators	Codes
stabilizer group $S = \langle g_1, \ldots, g_{n-k} \rangle$	linear binary code of length $2n$ $C_{bin} = \left[\begin{array}{c\|c} g_1^X & g_1^Z \\ \vdots & \\ g_{n-k}^X & g_{n-k}^Z \end{array}\right] = [\, X \mid Z \,]$ additive code over $GF(4)$ $C = [\, X + \omega \cdot Z \,]$
normalizer group $\mathcal{N}$ $\mathcal{N} = \{ \pi \in \mathcal{P}_n : \pi \cdot s = s \cdot \pi \ \forall s \in S \}$ all Pauli operators that commute with all elements of S $\Rightarrow$ they stabilize the code space, but not necessarily all vectors in the space	symplectic dual code $C^* = \{ v \in GF(4)^n : v * c = 0 \text{ for all } c \in C \}$ $C^*_{bin} = [\, X' \mid Z' \,]$ such that $X \cdot Z'^t - Z \cdot X'^t = 0$

„logical operations" on the code space

for the code C we have $C \leq C^*$

$\dim(C)=n-k$ $\qquad$ $\dim(D)=n$ $\qquad$ $\dim(C^*)=n+k$

$$C < \langle C, v_1\rangle < \ldots \quad D = D^* \quad \ldots < \langle C, v_1\rangle^* < C^*$$

$v_1 \in C^*$, $v_2 \in \langle C, v_1\rangle^*$

$$D < \langle D, w_1\rangle < \ldots < C^*$$

$$\begin{array}{r} n-k\{ \\ \\ k\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. \\ k\left\{\begin{array}{c} \\ \\ \end{array}\right. \end{array}\left[\begin{array}{c|c} X & Z \\ \hline v_1^X & v_1^Z \\ v_2^X & v_2^Z \\ v_k^X & v_k^Z \\ \hdashline w_1^X & w_1^Z \\ \vdots & \\ w_k^X & w_k^Z \end{array}\right] = \left[\begin{array}{c|c} S^X & S^Z \\ \hline \multicolumn{2}{c}{\overline{Z}_1} \\ \multicolumn{2}{c}{\vdots} \\ \multicolumn{2}{c}{\overline{Z}_k} \\ \hdashline \multicolumn{2}{c}{\overline{X}_1} \\ \multicolumn{2}{c}{\vdots} \\ \multicolumn{2}{c}{\overline{X}_k} \end{array}\right]$$

- The vectors v_i have ~~vanishing~~ vanishing sympl. inner product with all vectors of the code C and all other vectors v_j
  $\Rightarrow$ The operators $\overline{Z}_i$ are in the normalizer and they mutually commute.
- The vectors w_ℓ can be choosen such that the operators $X_\ell \in N$, X_ℓ and X_m commute, X_ℓ, Z_ℓ anti-commute, X_ℓ and Z_m commute for $\ell \neq m$

References

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