International School of Physics “Enrico Fermi”

Quantum Computers, Algorithms and Choas

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Quantum Error Correction

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Outline

- general setting of quantum error correction
- discretizing errors
- encoding circuits

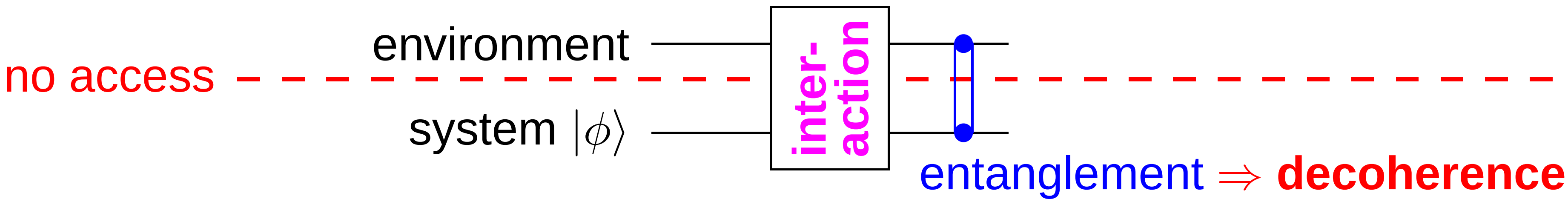
References

Markus Grassl, “Algorithmic aspects of quantum error-correcting codes”, in: Ranee K. Brylinski and Goong Chen (Eds.), *Mathematics of Quantum Computation*, Chapman & Hall/CRC, 2002, pp. 223-252.

Markus Grassl, Martin Rötteler, and Thomas Beth, “Efficient Quantum Circuits for Non-Qubit Quantum Error-Correcting Codes”, *International Journal of Foundations of Computer Science (IJFCS)*, Vol. 14, No. 5 (2003), pp. 757-775. Preprint quant-ph/0211014.

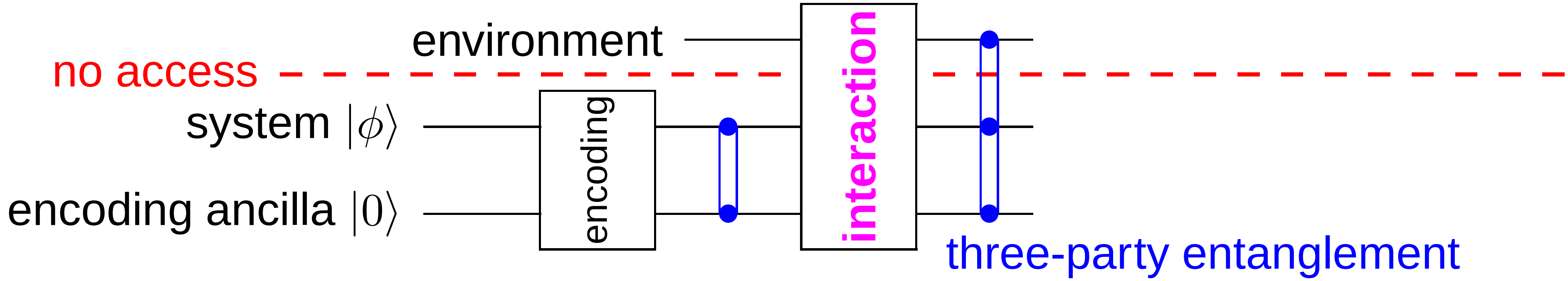
Quantum Error Correction

General scheme



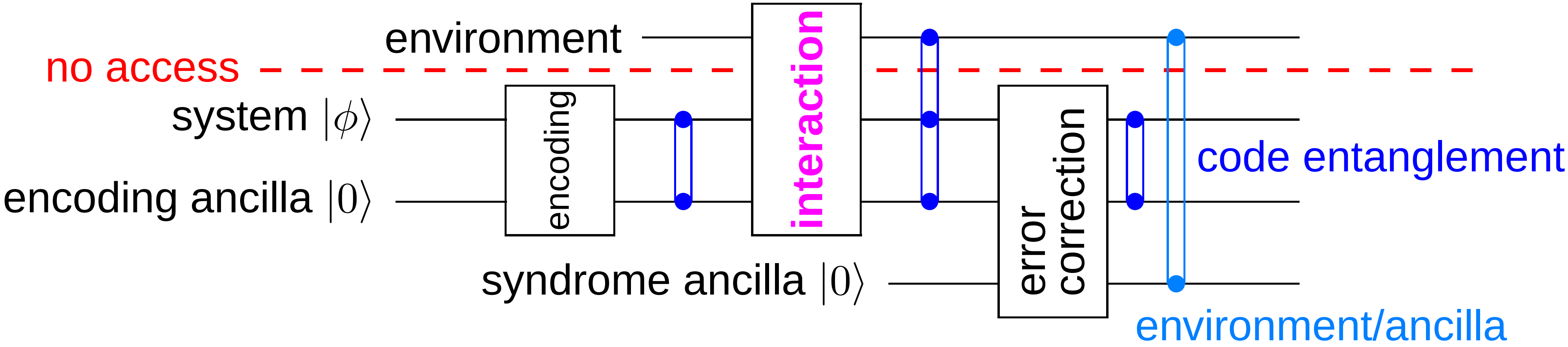
Quantum Error Correction

General scheme



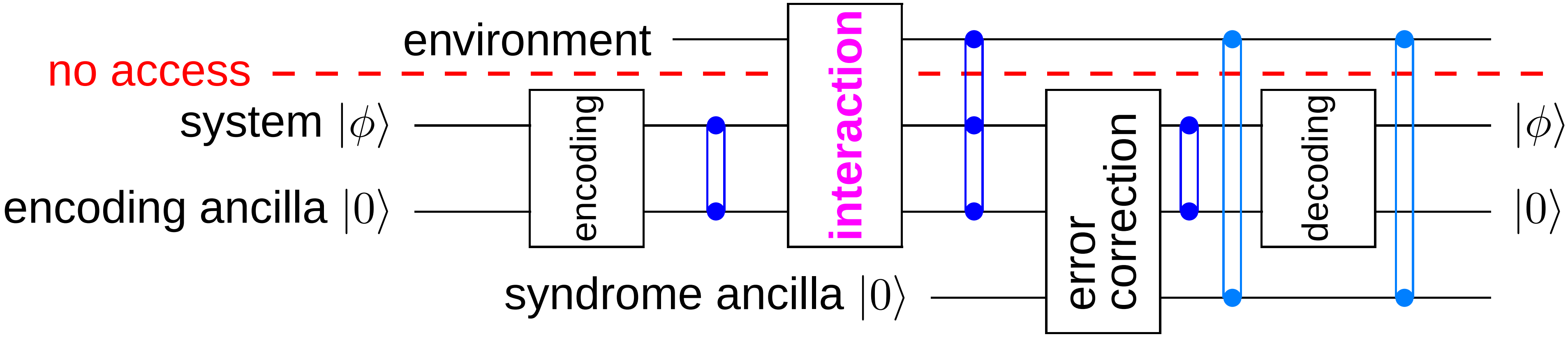
Quantum Error Correction

General scheme



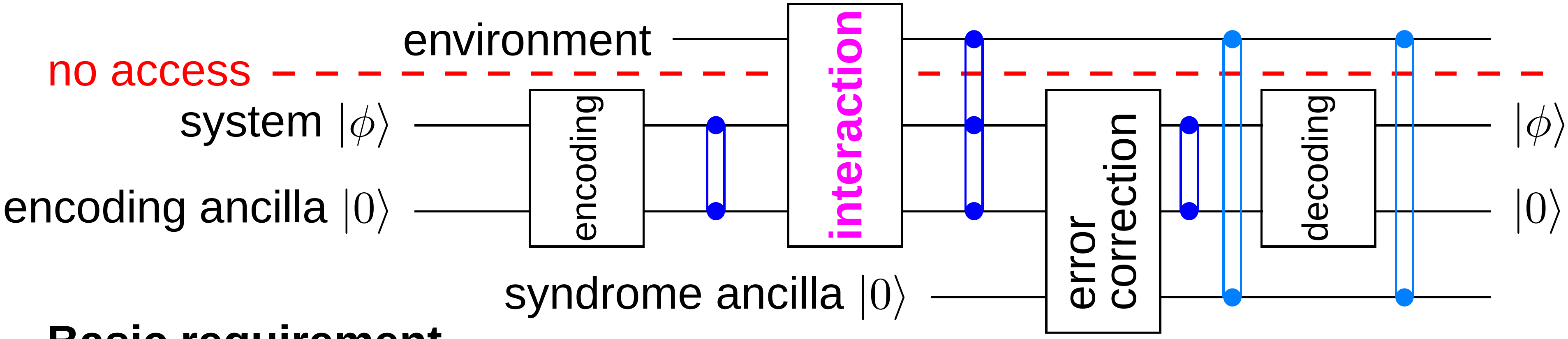
Quantum Error Correction

General scheme



Quantum Error Correction

General scheme



Basic requirement

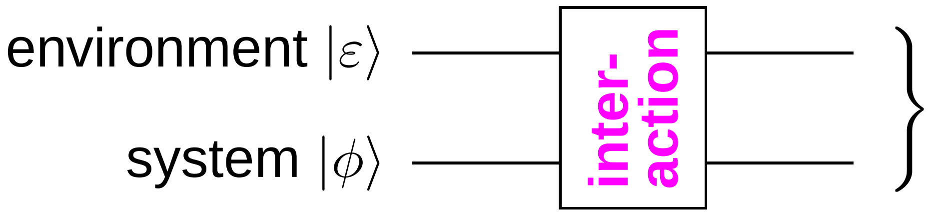
some knowledge about the interaction between system and environment

Common assumptions

- no initial entanglement between system and environment
- local or uncorrelated errors, i. e., only a few qubits are disturbed
  $\Longrightarrow$ CSS codes, stabilizer codes
- interaction with symmetry $\Longrightarrow$ decoherence free subspaces

Interaction System/Environment

“Closed” System



$$= U_{\text{env/sys}}\big(|\varepsilon\rangle|\phi\rangle\big)$$

“Channel”

$$\mathsf{Q}\colon\ \rho_{\text{in}} := |\phi\rangle\langle\phi| \longmapsto \rho_{\text{out}} := \mathsf{Q}(|\phi\rangle\langle\phi|) := \sum_i E_i \rho_{\text{in}} E_i^\dagger$$

with Kraus operators (error operators) E_i

Local/low correlated errors

- product channel $\mathsf{Q}^{\otimes n}$ where Q is “close” to identity
- Q can be expressed (approximated) with error operators $\tilde{E}_i$ such that each E_i acts on few subsystems, e. g. quantum gates

Computer Science Approach: *Discretize*

QECC Characterization

[Knill & Laflamme, PRA **55**, 900–911 (1997)]

A subspace $\mathcal{C}$ of $\mathcal{H}$ with orthonormal basis $\{|c_1\rangle, \ldots, |c_K\rangle\}$ is an error-correcting code for the error operators $\mathcal{E} = \{E_1, E_2, \ldots\}$, if there exists constants $\alpha_{k,l} \in \mathbb{C}$ such that for all $|c_i\rangle$, $|c_j\rangle$ and for all $E_k, E_l \in \mathcal{E}$:

$$\langle c_i|E_k^\dagger E_l|c_j\rangle = \delta_{i,j}\alpha_{k,l}. \tag{1}$$

It is sufficient that (1) holds for a vector space basis of $\mathcal{E}$.

Linearity of Conditions

Assume that $\mathcal{C}$ can correct the errors $\mathcal{E} = \{E_1, E_2, \ldots\}$.

New error-operators:

$$A := \sum_k \lambda_k E_k \quad \text{and} \quad B := \sum_l \mu_l E_l$$

$$\begin{aligned} \langle c_i | A^\dagger B | c_j \rangle &= \sum_{k,l} \overline{\lambda_k} \mu_l \langle c_i | E_k^\dagger E_l | c_j \rangle \\ &= \sum_{k,l} \overline{\lambda_k} \mu_l \delta_{i,j} \alpha_{k,l} \\ &= \delta_{i,j} \cdot \alpha'(A, B) \end{aligned}$$

It is sufficient to correct error operators that form a basis of the linear vector space spanned by the operators $\mathcal{E}$.

$\Longrightarrow$ only a finite set of errors.

Error Basis

Pauli Matrices

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- vector space basis of all 2×2 matrices
- unitary matrices which generate a *finite* group

Error Basis for many Qubits/Qudits

$\mathcal{E}$ error basis for subsystem of dimension d with $I \in \mathcal{E}$
$\Longrightarrow \mathcal{E}^{\otimes n}$ error basis with elements

$$E := E_1 \otimes \ldots \otimes E_n, \quad E_i \in \mathcal{E}$$

weight of E: number of factors $E_i \neq I$

Local Error Model

Code Parameters

$\mathcal{C} = [\![n, k, d]\!]$

n: number of subsystems used in total

k: number of (logical) subsystems encoded

d: “minimum distance”

– correct all errors acting on at most $(d-1)/2$ subsystems

– detect all errors acting on less than d subsystems

– correct all errors acting on less than d subsystems at *known* positions

General Decoding Algorithm

	$E_1\mathcal{C}$	$E_2\mathcal{C}$	$\cdots$	$E_k\mathcal{C}$	$\cdots$
$\mathcal{V}_0$	$E_1\|c_0\rangle$	$E_2\|c_0\rangle$	$\cdots$	$E_k\|c_0\rangle$	$\cdots$
$\mathcal{V}_1$	$E_1\|c_1\rangle$	$E_2\|c_1\rangle$	$\cdots$	$E_k\|c_1\rangle$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	
$\mathcal{V}_i$	$E_1\|c_i\rangle$	$E_2\|c_i\rangle$	$\cdots$	$E_k\|c_i\rangle$	$\cdots$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\ddots$

$$\langle c_i|E_k^\dagger E_l|c_j\rangle = \delta_{i,j}\alpha_{k,l} \tag{1}$$

General Decoding Algorithm

	$E_1\mathcal{C}$	$E_2\mathcal{C}$	$\cdots$	$E_k\mathcal{C}$	$\cdots$
$\mathcal{V}_0$	$E_1\|c_0\rangle$	$E_2\|c_0\rangle$	$\cdots$	$E_k\|c_0\rangle$	$\cdots$
$\mathcal{V}_1$	$E_1\|c_1\rangle$	$E_2\|c_1\rangle$	$\cdots$	$E_k\|c_1\rangle$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	
$\mathcal{V}_i$	$E_1\|c_i\rangle$	$E_2\|c_i\rangle$	$\cdots$	$E_k\|c_i\rangle$	$\cdots$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\ddots$

rows are orthogonal as $\langle c_i|E_k^\dagger E_l|c_j\rangle = 0$ for $i \neq j$

$$\langle c_i|E_k^\dagger E_l|c_j\rangle = \delta_{i,j}\alpha_{k,l} \qquad (1)$$

General Decoding Algorithm

	$E_1\mathcal{C}$	$E_2\mathcal{C}$	$\cdots$	$E_k\mathcal{C}$	$\cdots$
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$\mathcal{V}_1$	$E_1\|c_1\rangle$	$E_2\|c_1\rangle$	$\cdots$	$E_k\|c_1\rangle$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	
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$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\ddots$

rows are orthogonal as $\langle c_i|E_k^\dagger E_l|c_j\rangle = 0$ for $i \neq j$

inner product between columns is constant as $\langle c_i|E_k^\dagger E_l|c_i\rangle = \alpha_{k,l}$

$$\langle c_i|E_k^\dagger E_l|c_j\rangle = \delta_{i,j}\alpha_{k,l} \tag{1}$$

General Decoding Algorithm

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$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	
$\mathcal{V}_i$	$E_1\|c_i\rangle$	$E_2\|c_i\rangle$	$\cdots$	$E_k\|c_i\rangle$	$\cdots$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\ddots$

rows are orthogonal as $\langle c_i|E_k^\dagger E_l|c_j\rangle = 0$ for $i \neq j$

inner product between columns is constant as $\langle c_i|E_k^\dagger E_l|c_i\rangle = \alpha_{k,l}$

$\Longrightarrow$ simultaneous Gram-Schmidt orthogonalization within the spaces $\mathcal{V}_i$

Orthogonal Decomposition

	$E'_1\mathcal{C}$	$E'_2\mathcal{C}$	$\cdots$	$E'_k\mathcal{C}$	$\cdots$
$\mathcal{V}_0$	$E'_1\|c_0\rangle$	$E'_2\|c_0\rangle$	$\cdots$	$E'_k\|c_0\rangle$	$\cdots$
$\mathcal{V}_1$	$E'_1\|c_1\rangle$	$E'_2\|c_1\rangle$	$\cdots$	$E'_k\|c_1\rangle$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	
$\mathcal{V}_i$	$E'_1\|c_i\rangle$	$E'_2\|c_i\rangle$	$\cdots$	$E'_k\|c_i\rangle$	$\cdots$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\ddots$

rows are mutually orthogonal

columns are mutually orthogonal

- new error operators E'_k are linear combinations of the E_l
- projection onto $E'_k\mathcal{C}$ determines the error
- exponentially many orthogonal spaces $E'_k\mathcal{C}$

Encoding Stabilizer Codes

[Grassl, Rötteler, and Beth, IJFCS, **14** (2003), pp. 757-775]

Basic idea:

Use operations of the *generalized Clifford group* (or Jacobi group) to transform the stabilizer S into a trivial stabilizer $S_0 := \left\langle Z^{(1)}, \ldots Z^{(n-k)} \right\rangle$.

- row/column operations on the binary matrix $(X|Z)$ to obtain "normal form" $(0|I0)$
- operations on $(X|Z)$ correspond to
  - "elementary" single-qubit gates
  - CNOT-gate
  - single qudit gate $P := \begin{pmatrix} 1 & 0 \\ 0 & e^{\pi i/2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \in \mathbb{C}^{2\times 2}$

Action on Pauli Matrices

Hadamard matrix H	$HXH = Z$ $(1,0) \mapsto (0,1)$	$HYH = -Y$ $(1,1) \mapsto (1,1)$	$HZH = X$ $(0,1) \mapsto (1,0)$
	exchange X and Z		
matrix P	$P^\dagger XP = -Y$ $(1,0) \mapsto (1,1)$	$P^\dagger YP = X$ $(1,1) \mapsto (1,0)$	$P^\dagger ZP = Z$ $(0,1) \mapsto (0,1)$
	multiply X by Z		

in $\mathbb{C}$

$\bmod 2$

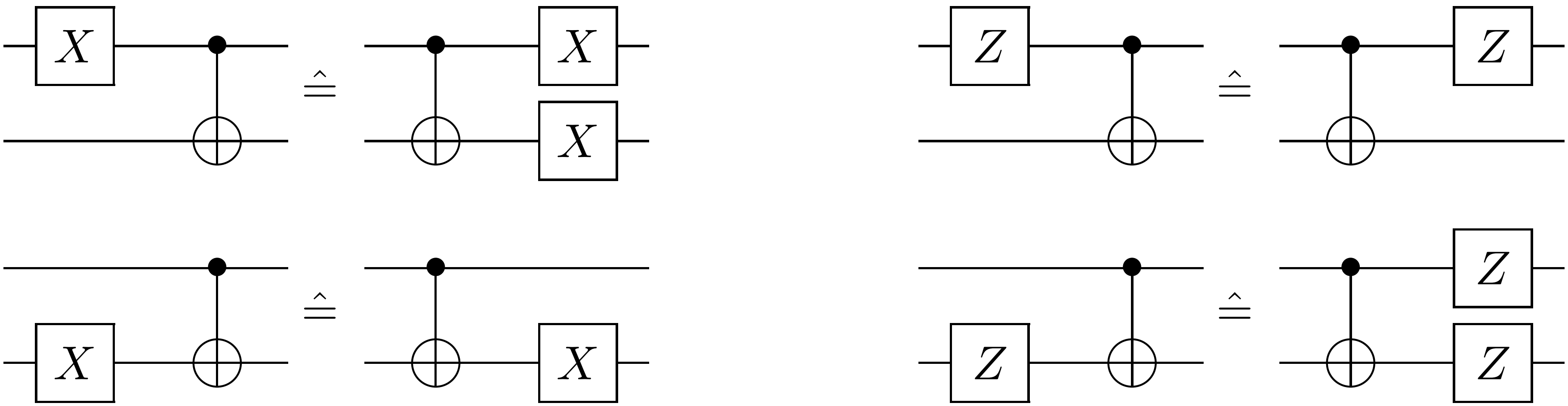
operation on binary row vectors: $(a,b)\widetilde{M} = (a',b')$ (arithmetic $\bmod 2$)

$$\widetilde{H} \mathrel{\hat{=}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \widetilde{P} \mathrel{\hat{=}} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

local operation on $(X|Z)$:
multiplying column i in submatrix X and column i in submatrix Z by $\widetilde{M}$

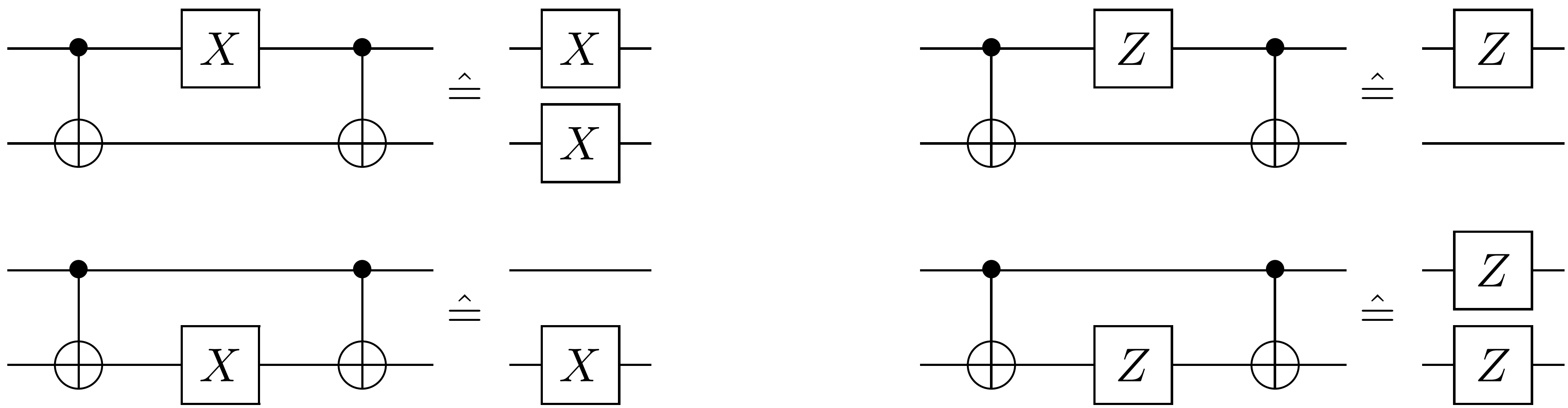
Action of CNOT

Error propagation



Action of CNOT

Modifying stabilizers



Action of CNOT

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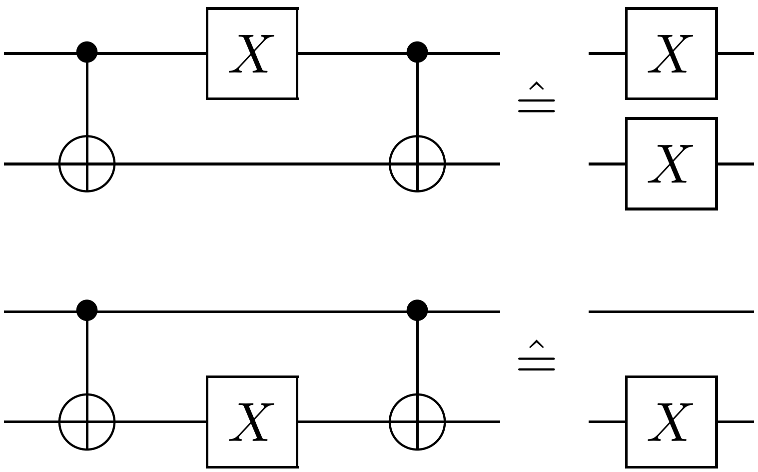
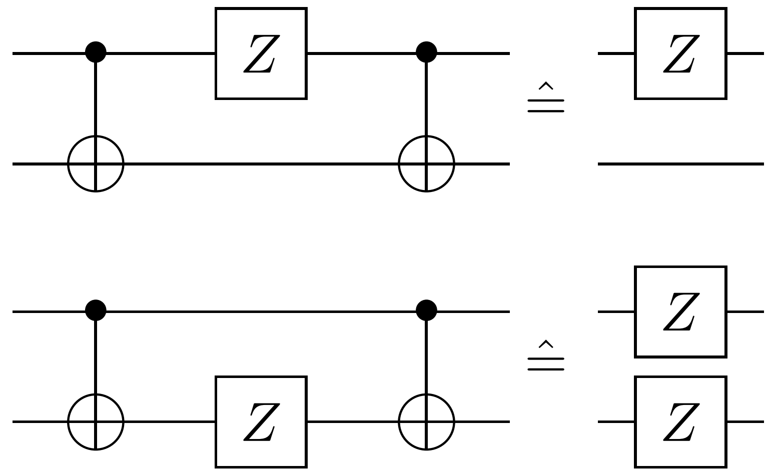


add X from source to target



add Z from target to source

Example: 5 Qubit Code $[\![5,1,3]\!]$

Generators of stabilizer

$$\begin{bmatrix} XXZIZ \\ ZXXZI \\ IZXXZ \\ ZIZXX \end{bmatrix} \hat{=} \left(\begin{array}{ccccc|ccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \end{array}\right)$$

Step I: X-Only Generator

Local operations $\hat{=}$ operations on corresponding X and Z columns

$$\left(\begin{array}{ccccc|ccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \end{array}\right)$$

$$T_1 := I \otimes I \otimes I \otimes I \otimes I$$

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Local operations $\hat{=}$ operations on corresponding X and Z columns

$$\left(\begin{array}{ccccc|ccccc} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$T_1 := I \otimes I \otimes H \otimes I \otimes I$$

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$$\left(\begin{array}{ccccc|ccccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$T_1 := I \otimes I \otimes H \otimes I \otimes H$$

Step II: X-Generator of Weight One

$\mathrm{CNOT} \mathrel{\hat{=}}$ operations on pairs of corresponding X and Z columns

$$\left(\begin{array}{ccccc|ccccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$T_1 := I \otimes I \otimes H \otimes I \otimes H$$

$$T_2 := I$$

Step II: X-Generator of Weight One

$\mathrm{CNOT} \mathrel{\hat{=}}$ operations on pairs of corresponding X and Z columns

$$\left(\begin{array}{ccccc|ccccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$T_1 := I \otimes I \otimes H \otimes I \otimes H$$

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Step II: X-Generator of Weight One

$\mathrm{CNOT} \hat{=}$ operations on pairs of corresponding X and Z columns

$$\left(\begin{array}{ccccc|ccccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$T_1 := I \otimes I \otimes H \otimes I \otimes H$$

$$T_2 := \mathrm{CNOT}^{(1,2)}$$

Step II: X-Generator of Weight One

$\mathrm{CNOT} \mathrel{\hat{=}}$ operations on pairs of corresponding X and Z columns

$$\left(\begin{array}{ccccc|ccccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$T_1 := I \otimes I \otimes H \otimes I \otimes H$$

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$$\left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$T_1 := \; I \otimes I \otimes H \otimes I \otimes H$$

$$T_2 := \mathrm{CNOT}^{(1,2)}\mathrm{CNOT}^{(1,3)}$$

Step II: X-Generator of Weight One

$\mathrm{CNOT} \mathrel{\hat{=}}$ operations on pairs of corresponding X and Z columns

$$\left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & {\color{red}1} & 0 & 0 & 0 & 0 & {\color{blue}0} \\ 0 & 1 & 0 & 0 & {\color{red}0} & 0 & 0 & 1 & 1 & {\color{blue}0} \\ 0 & 0 & 0 & 1 & {\color{red}1} & 0 & 1 & 1 & 0 & {\color{blue}0} \\ 0 & 0 & 1 & 1 & {\color{red}0} & 1 & 0 & 0 & 0 & {\color{blue}1} \end{array}\right)$$

$$T_1 := I \otimes I \otimes H \otimes I \otimes H$$

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Step II: X-Generator of Weight One

$\mathrm{CNOT} \hat{=}$ operations on pairs of corresponding X and Z columns

$$\left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$T_1 := I \otimes I \otimes H \otimes I \otimes H$$

$$T_2 := \mathrm{CNOT}^{(1,2)}\mathrm{CNOT}^{(1,3)}\mathrm{CNOT}^{(1,5)}$$

Step III: Row Operations

multiplying generators $\hat{=}$ adding/permuting rows

$$\left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$T_1 := I \otimes I \otimes H \otimes I \otimes H$$

$$T_2 := \mathrm{CNOT}^{(1,2)}\mathrm{CNOT}^{(1,3)}\mathrm{CNOT}^{(1,5)}$$

Step III: Row Operations

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$$\left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \end{array}\right)$$

$$T_1 := I \otimes I \otimes H \otimes I \otimes H$$

$$T_2 := \mathrm{CNOT}^{(1,2)}\mathrm{CNOT}^{(1,3)}\mathrm{CNOT}^{(1,5)}$$

Step IV: Next Row/Generator

$$\left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \end{array}\right)$$

$$T_3 := I \otimes I \otimes I \otimes I \otimes I$$

Step IV: Next Row/Generator

$$\left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right)$$

$$T_3 := I \otimes I \otimes H \otimes H \otimes I$$

Step IV: Next Row/Generator

$$\left(\begin{array}{ccccc|ccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0
\end{array}\right)$$

$$T_3 := I \otimes I \otimes H \otimes H \otimes I$$

$$T_4 := \mathrm{CNOT}^{(2,3)}\mathrm{CNOT}^{(2,4)}$$

Step IV: Next Row/Generator

$$\left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right)$$

$$T_3 := I \otimes I \otimes H \otimes H \otimes I$$

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Pre-Final Result: Only X-Generators

$$\left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Final Result: Only Z-Generators

$$\left(\begin{array}{ccccc|ccccc} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$T_{\text{final}} := H \otimes H \otimes H \otimes H \otimes I$$

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$$T_{\text{final}} := H \otimes H \otimes H \otimes H \otimes I$$

combining all transformations: quantum circuit that maps encoded state $|\psi_L\rangle$ to the un-encoded state $|0\rangle|\psi\rangle$

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$$T_{\text{final}} := H \otimes H \otimes H \otimes H \otimes I$$

combining all transformations: quantum circuit that maps encoded state $|\psi_L\rangle$ to the un-encoded state $|0\rangle|\psi\rangle$

“Homework”: derive the complete circuit

Decoding Circuit for Ternary QECC $\mathcal{C} = [\![9,5,3]\!]_3$