Lecture 11: CSS Codes

Preliminaries for Stabilizer Codes

11.1 Shor's Nine Qubit Code

two levels of encoding based on the following codes

$\mathcal{C}_1$: $|0\rangle \mapsto |000\rangle$

$|1\rangle \mapsto |111\rangle$ correcting one σ_x-error

CSS code with $C_1 = [3,1,3]$

$C_2 = [3,3,1]$, $C_2^\perp = [3,0]$

$$|\psi_i\rangle = \sum_{c \in C_2^\perp} |c + w_i\rangle \qquad w_i \in C_1 / C_2^\perp$$

$\mathcal{C}_2$: a) $|0\rangle \mapsto |{+}{+}{+}\rangle$

$|1\rangle \mapsto |{-}{-}{-}\rangle$ correcting one σ_z-error

where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = H|0\rangle$

$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = H|1\rangle$

$$H|\psi_i\rangle = \sum_{c \in C_2} (-1)^{c \cdot w_i} |c\rangle$$

b) $|0\rangle \mapsto \frac{1}{2}\left(|000\rangle + |011\rangle + |101\rangle + |110\rangle\right)$

$|1\rangle \mapsto \frac{1}{2}\left(|001\rangle + |010\rangle + |100\rangle + |111\rangle\right)$

$$|\phi_i\rangle = \sum_{c \in C_2^{\prime\perp}} |c + w_i\rangle$$

$[n, k, d]$ $\qquad G \in \mathbb{F}_2^{k \times n}$

$C_2^\perp \leq C_1$

$\Leftrightarrow C_1^\perp \leq C_2$

$C_2^{\prime\perp} = [3, 2, 2]$ generated by $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

$C_2^{\prime} = [3, 1, 3]$ generated by $(1 1 1)$

$C_1^{\prime} = [3, 3, 1]$

Component codes are CSS codes

first level: $|\underline{0}\rangle := |000\rangle$

$|\underline{1}\rangle := |111\rangle$

$(Z \otimes I \otimes I)\,|000\rangle = |\underline{0}\rangle$ | $(Z I I)\,|111\rangle = -|111\rangle = -|\underline{1}\rangle$

$(I \otimes Z \otimes I)\,|000\rangle = |\underline{0}\rangle$ | $(I Z I)\,|111\rangle = -|111\rangle = -|\underline{1}\rangle$

$(I \otimes I \otimes Z)\,|000\rangle = |\underline{0}\rangle$ | $(I I Z)\,|111\rangle = -|111\rangle = -|\underline{1}\rangle$

$\Rightarrow$ single σ_z-errors on the physical level act as „encoded Z-operators" on the „logical qubits" $|\underline{0}\rangle$ and $|\underline{1}\rangle$

encoding to protect against these logical Z-errors:

$$|\underline{0}\rangle \mapsto |\underline{+}\,\underline{+}\,\underline{+}\rangle$$

$$|\underline{1}\rangle \mapsto |\underline{-}\,\underline{-}\,\underline{-}\rangle$$

$$|\underline{0}\rangle \mapsto (|\underline{0}\rangle + |\underline{1}\rangle)(|\underline{0}\rangle + |\underline{1}\rangle)(|\underline{0}\rangle + |\underline{1}\rangle)$$
$$= (|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$

$$|\underline{1}\rangle \mapsto (|\underline{0}\rangle - |\underline{1}\rangle)(|\underline{0}\rangle - |\underline{1}\rangle)(|\underline{0}\rangle - |\underline{1}\rangle)$$
$$= (|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$$

The code can correct up to three σ_x-errors, if there is only error per 3-qubit block.
It can also correct 2 σ_z errors if they are within the same block
$\Rightarrow$ ~~All~~ errors of weight one plus some specific errors of higher weight.

11.2 Encoding / Error Correction for CSS codes

ingredients: binary linear block codes

$$C_1 = [n, k_1, d_1]$$

$$C_2 = [n, k_2, d_2]$$

with $C_2^\perp \leq C_1$

$\{ w_i : i = 1, .., K = 2^{k_1+k_2-n} \}$ a set

of coset representatives of $C_1 / C_2^\perp$

basis states: $$|\psi_i\rangle = \sum_{c \in C_2^\perp} | c + w_i \rangle$$

general state: $$|\psi\rangle = \sum_i \alpha_i |\psi_i\rangle = \sum_{c \in C_1} \beta_c | c \rangle$$

general error: $$e := \left(\sigma_x^{e_{x,1}} \sigma_z^{e_{z,1}}\right) \otimes \ldots \otimes \left(\sigma_x^{e_{x,n}} \sigma_z^{e_{z,n}}\right)$$

$$\hat{=} (e_x | e_z) \in \mathbb{F}_2^{2 \cdot n}$$

$$e|\psi\rangle = \sum_{c \in C_1} \beta_c (-1)^{c \cdot e_z} | c + e_x \rangle$$

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generator and parity check matrices of C_1, C_2

recall: The parity check matrix of a code C is a generator matrix for the dual code $C^\perp$.

$$C_2^\perp \le C_1$$

generator matrix for C_1: $G_1 = \left[\begin{array}{c} H_2 \\ \hline D_1 \end{array}\right]$

H_2 is parity check matrix of C_2

D_1 generates the vector space complement of $C_2^\perp$ in C_1

similar:

$$G_2 = \left[\begin{array}{c} H_1 \\ \hline D_2 \end{array}\right]$$

$$e|\psi\rangle = \sum_{c\in C_1} \beta_c (-1)^{c\cdot e_z} |c+e_x\rangle |0\rangle$$

$$\downarrow$$

$$\Big(\sum_{c\in C_1} \beta_c (-1)^{c\cdot e_z} \underbrace{|c+e_x\rangle}_{n \text{ qubits}}\Big) \underbrace{|e_x\cdot H_1^t\rangle}_{n-k_1 \text{ qubits}}$$

$= (c+e_x)\cdot H_1^t$

$$e|\psi\rangle|0\rangle \mapsto (e|\psi\rangle) \otimes |e_x\cdot H_1^t\rangle$$

$$\mapsto (e|\psi\rangle) \otimes |e_x\cdot H_1^t\rangle \otimes |e_z\cdot H_2^t\rangle$$

11.3 Decoding of CSS-Codes using measurements

1. Use the transformation

$$S_1 : |x\rangle|y\rangle \mapsto |x\rangle|\underbrace{y + x \cdot H_1^t}_{s_1}\rangle$$

to compute an $n-k_1$ (qu-)bit error syndrome of the σ_x errors.

2. Perform a Hadamard transformation on the first n qubits
3. Use the transformation

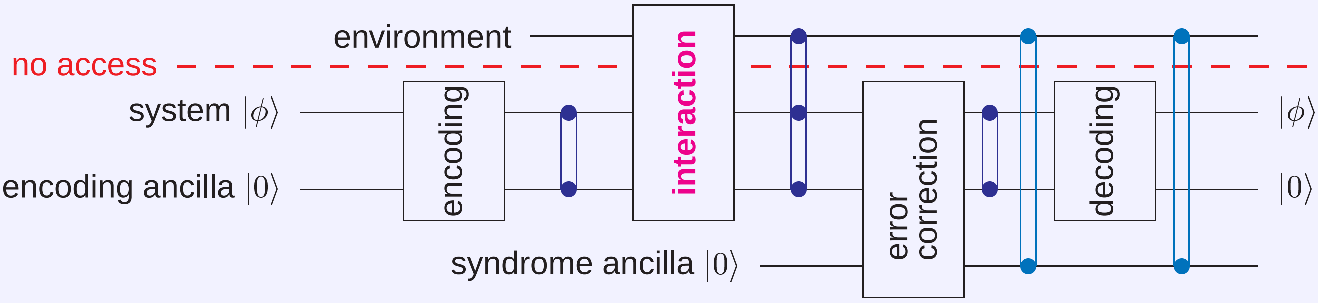
$$S_2 : |x\rangle|y\rangle \mapsto |x\rangle|\overbrace{y + x \cdot H_2^t}^{s_2}\rangle$$

to compute an $n-k_2$ (qu-)bit error syndrome of the σ_z errors.

4. Perform a Hadamard transformation on the first n qubits.
5. Measure the error syndromes s_1 and s_2
6. Use classical decoding algorithms for the codes C_1 and C_2 to compute error vectors e_x and e_z corresponding to the syndromes s_1 and s_2, resp.
7. Correct the errors using $\sigma_x^{e_x} \cdot \sigma_z^{e_z}$.

Quantum Error-Correction

General scheme



Basic requirement

some knowledge about the interaction between the system and the environment

Common assumptions

- no initial entanglement between system and environment
- local or uncorrelated errors, i. e., only a few qubits are disturbed
  $\Longrightarrow$ CSS codes, stabilizer codes
- interaction with symmetry
  $\Longrightarrow$ decoherence free subspaces/subsystems