Isaac Newton Institute for Mathematical Sciences
Complexity, Computation and the Physics of Information
Workshop: Quantum Computation and Algorithms

Description of Multi-Particle Entanglement through Polynomial Invariants

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Outline

- Statement of the Problem
- Motivation
- Our Approach:
  Ring of Polynomial Invariants
- Examples
- Molien Series
- Warm-up: Two Particles
- Generalised Molien Series
- Results: Three Qubits
- Outlook: Four Qubits

Main Problem

Given two quantum states

$$|\psi\rangle \quad \text{and} \quad |\phi\rangle$$

on n particles (qubits)
is there a local unitary transformation

$$U = U_1 \otimes U_2 \otimes \ldots \otimes U_n$$

with $U|\psi\rangle = |\phi\rangle$?

Related Work

Rains	quant-ph/9704042
Grassl, Rötteler, Beth	PRA 58, pp. 1833-1839 (1998)
	quant-ph/9712040
Linden, Popescu	quant-ph/9711016
Linden, Popescu, Sudbery	quant-ph/9801076
Nielsen	quant-ph/9811053
Vidal	quant-ph/9902033

and many other results & definitions for multi-particle entanglement

Motivation (I): Bell Basis

$$
\begin{aligned}
|\Phi^+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
|\Phi^-\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\
|\Psi^+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\
|\Psi^-\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)
\end{aligned}
$$

- Same non–local properties
- Related by local transformations $U_1 \otimes U_2$
- Schmidt decomposition:

$$
|\psi\rangle = \alpha|\uparrow_1\uparrow_2\rangle + \beta|\downarrow_1\downarrow_2\rangle
$$

with local orthogonal basis states $|\uparrow_i\rangle$, $|\downarrow_i\rangle$

Problem:

Generalization for more than two subsystems?

Motivation (II): Description of Entanglement

- Independent of local basis
- Various definitions exist
- Partitioning of more than two subsystems into groups

Problems:

- Check whether two quantum states are equivalent with respect to local unitary transformations.
- Describe the equivalence classes of quantum states, i. e., the orbits under local unitary transformations.

Motivation (III): Quantum Codes

- Error operators:
  tensor products of $\mathbb{1}$, σ_x, σ_y, and σ_z
- Weight of errors:
  number of tensor factors $\neq \mathbb{1}$
- The weight doesn't change under local unitary transformations

$$U = U_1 \otimes \ldots \otimes U_N \quad \text{where } U_i \in U(2)$$

Problem:

Check whether two codes are equivalent with respect to local unitary transformations.

Local Polynomial Invariants

Use the polynomial invariants of the groups

- $SU(2) \otimes \ldots \otimes SU(2)$
- $U(2) \otimes \ldots \otimes U(2)$

operating on

- pure states $|\psi\rangle$
- mixed states ρ

to describe multi-particle entanglement.

Operation of $GL(N, \mathbb{F})$

Linear operation

on polynomials $f \in \mathbb{F}[x_1, \ldots, x_N] =: \mathbb{F}[\boldsymbol{x}]$

$$f(\boldsymbol{x})^g := f(\boldsymbol{x}^g) \quad \text{where} \quad \boldsymbol{x}^g = (x_1, \ldots, x_N) \cdot g$$

$\Rightarrow$ pure quantum states

Operation by conjugation

on polynomials $f \in \mathbb{F}[x_{11}, \ldots, x_{NN}] =: \mathbb{F}[X]$

$$f(X)^g := f(X^g) \quad \text{where}$$

$$X^g = g^{-1} \cdot \begin{pmatrix} x_{11} & \cdots & x_{1N} \\ \vdots & \ddots & \vdots \\ x_{N1} & \cdots & x_{NN} \end{pmatrix} \cdot g$$

$\Rightarrow$ mixed quantum states

Example: Invariants of S_3

$$S_3 = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$$

Power sums

$$\begin{aligned} p_1 &= x_1 + x_2 + x_3 \\ p_2 &= x_1^2 + x_2^2 + x_3^2 \\ p_3 &= x_1^3 + x_2^3 + x_3^3 \end{aligned}$$

Elementary symmetric polynomials

$$\begin{aligned} s_1 &= x_1 + x_2 + x_3 \\ s_2 &= x_1x_2 + x_2x_3 + x_3x_1 \\ s_3 &= x_1x_2x_3 \end{aligned}$$

Any invariant of S_3 can be expressed uniquely as a polynomial in p_1, p_2, p_3 (or s_1, s_2, s_3).

Example: Invariants of Z_3

$$Z_3 = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle$$

- $\mathbb{F}[x_1, x_2, x_3]^{S_3} \subseteq \mathbb{F}[x_1, x_2, x_3]^{Z_3}$ since $Z_3 \leq S_3$
- additional invariant of degree 3:

$$f_4 = x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_1$$

- s_1, s_2, s_3, and f_4 are not algebraically independent:

$$f_4^2 + (3s_3 - s_1 s_2) f_4 \\ +s_1^3 s_3 + s_2^3 - 6 s_1 s_2 s_3 + 9 s_3^2 = 0$$

- f_4 is not redundant:

$$\begin{aligned} f_4(2,3,4) &= 82 \\ f_4(2,4,3) &= 80 \end{aligned}$$

Ring of Invariants

Basic problem:

Given a subgroup $G \leq GL(N, \mathbb{F})$, which polynomials in N (or N^2) variables are invariant under linear operation (or operation by conjugation)?
Notation:

$$\mathbb{F}[\boldsymbol{x}]^G := \{f(\boldsymbol{x}) \in \mathbb{F}[\boldsymbol{x}] | \forall g \in G : f(\boldsymbol{x})^g = f(\boldsymbol{x})\}$$

Properties of $\mathbb{F}[\boldsymbol{x}]^G$:

- Homogeneous polynomials remain homogeneous
  $\Rightarrow$ consider only homogeneous polynomials.
- Any linear combination of invariants is an invariant.
- The product of invariants is an invariant.
- For reductive groups $\mathbb{F}[\boldsymbol{x}]^G$ is finitely generated.

Molien Series

- Formal power series with non–negative integer coefficients
- Encodes the vector space dimension d_k of the homogeneous invariants of degree k:

$$M(z) := \sum_{k \geq 0} d_k z^k \in \mathbb{Z}[\![z]\!].$$

- A rational function (for finitely generated algebras)
- General formula (for linear operation)

$$M(z) = \int_{g \in G} \mathrm{d}\mu_G(g) \frac{1}{\det(id - z \cdot g)}$$

Problems:

1. Applies only for linear operation
2. Integral is very difficult to compute

Two Particles

Pure State
$$|\psi\rangle = \sum_{i=1}^{d^2} x_i |b_i\rangle$$

with a (global) orthonormal basis $|b_i\rangle$.

Schmidt decomposition
$$|\psi\rangle = \sum_{j=1}^{d} \alpha_j |b_i^{(1)}\rangle |b_j^{(2)}\rangle$$

with local orthonormal bases $|b_j^{(1)}\rangle$, and $|b_j^{(2)}\rangle$.

Invariants The (real) coefficients α_j are the local invariants.

Problem The invariants α_j are no polynomial function in the coefficients x_i.

Example: Two Qubits

Pure State

$$|\psi\rangle = x_{00}|00\rangle + x_{01}|01\rangle + x_{10}|10\rangle + x_{11}|11\rangle$$

Invariants

$$\begin{aligned}
\operatorname{tr}(|\psi\rangle\langle\psi|) &= x_{00}\overline{x}_{00} + x_{01}\overline{x}_{01} + x_{10}\overline{x}_{10} + x_{11}\overline{x}_{11} \\
\operatorname{tr}\left((\operatorname{tr}_i |\psi\rangle\langle\psi|)^2\right) &= x_{00}^2\overline{x}_{00}^2 + x_{01}^2\overline{x}_{01}^2 + x_{10}^2\overline{x}_{10}^2 + x_{11}^2\overline{x}_{11}^2 \\
&\quad +2x_{00}x_{01}\overline{x}_{00}\overline{x}_{01} + 2x_{00}x_{10}\overline{x}_{00}\overline{x}_{10} + 2x_{00}x_{11}\overline{x}_{01}\overline{x}_{10} \\
&\quad +2x_{01}x_{10}\overline{x}_{00}\overline{x}_{11} + 2x_{01}x_{11}\overline{x}_{01}\overline{x}_{11} + 2x_{10}x_{11}\overline{x}_{10}\overline{x}_{11}
\end{aligned}$$

Problem We need complex conjugated variables.

Generalized Molien Series

Bi-degree of polynomials f in variables x_i and $\overline{x}_i$:

$$(\deg_{x_1,\ldots,x_n} f,\ \deg_{\overline{x}_1,\ldots,\overline{x}_n} f)$$

F-Series (Michael Forger)†

- Formal power series with non–negative integer coefficients
- Encodes the vector space dimension $d_{k,\ell}$ of the homogeneous invariants of bi-degree (k,ℓ):

$$F(z,\overline{z}) := \sum_{k,\ell\geq 0} d_{k,\ell}\, z^k\, \overline{z}^\ell \in \mathbb{Z}[\![z,\overline{z}]\!].$$

- General formula (for linear operation)

$$F(z,\overline{z}) = \int_G \mathrm{d}\mu_G(g) \frac{1}{\det(id - z\cdot g)} \frac{1}{\det(id - \overline{z}\cdot \overline{g})}$$

† J. Math. Phys. 39, pp. 1107–1141 (1998)

Two Qubits: F-Series of $SU(2) \otimes SU(2)$

$$F(\overline{z}, z) = \int_{U \in G} \mathrm{d}\mu_G(U) \frac{1}{\det(id - z \cdot U)} \frac{1}{\det(id - \overline{z} \cdot U^t)}$$

$$\vdots$$

$$= \oint_{\Gamma_w} \oint_{\Gamma_x} \frac{(1-w^2)(1-x^2)}{\prod\limits_{a,b \in \{1,-1\}} (1 - z \cdot w^a x^b)\,(1 - \overline{z} \cdot w^{-a} x^{-b})} \frac{\mathrm{d}w}{w} \frac{\mathrm{d}x}{x}$$

$$\vdots$$

$$= \frac{1}{(1 - z^2\overline{z}^2)(1 - \overline{z}^4)(1 - z^4)}$$

($G = SU(2) \otimes SU(2)$, $U = U_1 \otimes U_2$, Γ =complex unit circle)

Two Qubits: Molien Series of $U(2) \otimes U(2)$

$$
\begin{aligned}
F(\overline{z}, z) &= \frac{1}{(1 - z\overline{z})(1 - \overline{z}^2)(1 - z^2)} \\
&= 1 + z^2 + z\overline{z} + \overline{z}^2 + z^4 + z^3\overline{z} + 2z^2\overline{z}^2 + z\overline{z}^3 + \overline{z}^4 \\
&\quad + z^6 + z^5\overline{z} + 2z^4\overline{z}^2 + 2z^3\overline{z}^3 + 2z^2\overline{z}^4 + z\overline{z}^5 + \overline{z}^6 \\
&\quad + z^8 + z^7\overline{z} + 2z^6\overline{z}^2 + 2z^5\overline{z}^3 + 3z^4\overline{z}^4 \\
&\quad\quad + 2z^3\overline{z}^5 + 2z^2\overline{z}^6 + z\overline{z}^7 + \overline{z}^8 + \ldots
\end{aligned}
$$

Invariants of $U(2) \otimes U(2)$ must be invariant when multiplying with $e^{i\theta}$
$\Rightarrow$ Extract all coefficients of bi-degree (k, k)

$$
\begin{aligned}
M(z) &= 1 + z^2 + 2z^4 + 2z^6 + 3z^8 + \ldots \\
&= \frac{1}{(1 - z^2)(1 - z^4)}
\end{aligned}
$$

Three Qubits: Ansatz F-Series of $SU(2)^{\otimes 3}$

$$F(\overline{z}, z) = \int_{U \in G} \mathrm{d}\mu_G(U) \frac{1}{\det(id - z \cdot U)} \frac{1}{\det(id - \overline{z} \cdot U^t)}$$

$$= \frac{1}{(2\pi i)^3} \oint_{\Gamma_v} \oint_{\Gamma_w} \oint_{\Gamma_x} \frac{(1-v^2)(1-w^2)(1-x^2)}{\prod\limits_{a,b,c \in \{1,-1\}} (1 - z \cdot v^a w^b x^c)\,(1 - \overline{z} \cdot v^a w^b x^c)} \frac{\mathrm{d}v}{v} \frac{\mathrm{d}w}{w} \frac{\mathrm{d}x}{x}$$

($G = SU(2)^{\otimes 3}$, $U = U_1 \otimes U_2 \otimes U_3$, $\Gamma =$complex unit circle)

Computation of the integral using the theorem of residues

- Symbolic computation of singularities and residues
- Data type: factored rational functions implemented in MAGMA

Three Qubits: F-Series of $SU(2)^{\otimes 3}$

$$F(z,\overline{z}) = \frac{z^5\overline{z}^5 + z^3\overline{z}^3 + z^2\overline{z}^2 + 1}{(1-z\overline{z})(1-z^4)(1-\overline{z}^4)(1-z^2\overline{z}^2)^2(1-z\overline{z}^3)(1-z^3\overline{z})}$$

$$\begin{aligned}
&= 1 + z\overline{z} + z^4 + z^3\overline{z} + 4z^2\overline{z}^2 + z\overline{z}^3 + \overline{z}^4 \\
&\quad + z^5\overline{z} + z^4\overline{z}^2 + 5z^3\overline{z}^3 + z^2\overline{z}^4 + z\overline{z}^5 \\
&\quad + z^8 + z^7\overline{z} + 5z^6\overline{z}^2 + 5z^5\overline{z}^3 \\
&\quad + 12z^4\overline{z}^4 + 5z^3\overline{z}^5 + 5z^2\overline{z}^6 + z\overline{z}^7 + \overline{z}^8 \\
&\quad + z^9\overline{z} + z^8\overline{z}^2 + 6z^7\overline{z}^3 + 6z^6\overline{z}^4 + 15z^5\overline{z}^5 \\
&\qquad + z\overline{z}^9 + z^2\overline{z}^8 + 6z^3\overline{z}^7 + 6z^4\overline{z}^6 \\
&\quad + z^{12} + z^{11}\overline{z} + 5z^{10}\overline{z}^2 + 6z^9\overline{z}^3 + 16z^8\overline{z}^4 + 16z^7\overline{z}^5 + 30z^6\overline{z}^6 \\
&\qquad + \overline{z}^{12} + z\overline{z}^{11} + 5z^2\overline{z}^{10} + 6z^3\overline{z}^9 + 16z^4\overline{z}^8 + 16z^5\overline{z}^7 \\
&\quad + \ldots
\end{aligned}$$

Three Qubits: Molien Series of $U(2)^{\otimes 3}$

$$
\begin{aligned}
M(z) &= \frac{z^{12}+1}{(1-z^2)(1-z^4)^3(1-z^6)(1-z^8)} \\
&= 1 + z^2 + 4z^4 + 5z^6 + 12z^8 + 15z^{10} + 30z^{12} \\
&\quad + 37z^{14} + 65z^{16} + 80z^{18} + 128z^{20} + 156z^{22} \\
&\quad + 234z^{24} + 282z^{26} + 402z^{28} + 480z^{30} + \ldots
\end{aligned}
$$

Three Qubits: Invariant Ring of $SU(2)^{\otimes 3}$

Coefficient vector:

$$\boldsymbol{x} = \Big(\underbrace{x_{000}, x_{001}}_{00}, \underbrace{x_{010}, x_{011}}_{01}, \underbrace{x_{100}, x_{101}}_{10}, \underbrace{x_{110}, x_{111}}_{11}\Big)$$

Invariants of $\mathbb{1}_4 \otimes SU(2)$:

brackets $[i, j] := x_{i0}x_{j1} - x_{i1}x_{j0}$

inner products $\langle i, j\rangle := x_{i0}\overline{x}_{j0} + x_{i1}\overline{x}_{j1}$

Invariants of $SU(2) \otimes SU(2) \otimes SU(2)$:

permutations (π_1, π_2, π_3):

$$f_{\pi_1,\pi_2,\pi_3} = \sum_{i,j,\ldots} x_{i_1,i_2,i_3}\overline{x}_{\pi_1(i_1),\pi_2(i_2),\pi_3(i_3)} \cdot x_{j_1,j_2,j_3}\overline{x}_{\pi_1(j_1),\pi_2(j_2),\pi_3(j_3)} \cdot \ldots$$

Three Qubits: Invariant Ring of $SU(2)^{\otimes 3}$

Generators:

	bi-degree	permutations (π_1, π_2, π_3), brackets, inner products	#terms
f_1	$(1,1)$	(id, id, id)	8
f_2	$(2,2)$	$((1,2),(1,2),id)$	36
f_3	$(2,2)$	$((1,2),id,(1,2))$	36
s_1	$(4,0)$	$[1,2]^2 - 2[0,1][2,3] - 2[0,2][1,3] + [0,3]^2$	12
$\overline{s_1}$	$(0,4)$	$\overline{[1,2]}^2 - 2\overline{[0,1]}\,\overline{[2,3]} - 2\overline{[0,2]}\,\overline{[1,3]} + \overline{[0,3]}^2$	12
s_2	$(3,1)$	$[3,0]\langle 0,0\rangle - [3,0]\langle 3,3\rangle + [3,1]\langle 0,1\rangle + [3,2]\langle 0,2\rangle$ $+2[3,2]\langle 1,3\rangle - 2[1,0]\langle 2,0\rangle - [1,0]\langle 3,1\rangle - [2,0]\langle 3,2\rangle$ $-[2,1]\langle 0,0\rangle - [2,1]\langle 1,1\rangle + [2,1]\langle 2,2\rangle + [2,1]\langle 3,3\rangle$	40
$\overline{s_2}$	$(1,3)$		40
f_4	$(2,2)$	$(id,(1,2),(1,2))$	36
f_5	$(3,3)$	$((1,2),(2,3),(1,3))$	176
$f_4 f_5$	$(5,5)$		3760

Three Qubits: Invariant Ring of $U(2)^{\otimes 3}$

Generators of the invariant ring:

	degree	permutations (π_1, π_2, π_3)	#terms
f_1	2	(id, id, id)	8
f_2	4	$((1,2),(1,2),id)$	36
f_3	4	$((1,2),id,(1,2))$	36
f_4	4	$(id,(1,2),(1,2))$	36
f_5	6	$((1,2),(2,3),(1,3))$	176
f_6	8	$s_1\overline{s}_1$	144
f_7	12	$\overline{s}_1 s_2^2$	5988

$f_1,\ldots,f_6$ are algebraic independent. Relation for f_7:

$$f_7^2 + c_1(f_1,\ldots,f_6)f_7 + c_0(f_1,\ldots,f_6) \qquad \text{where } c_0, c_1 \in \mathbb{Q}[f_1,\ldots,f_6].$$

Four Qubits: Ansatz F-Series of $SU(2)^{\otimes 4}$

$$F(\overline{z}, z) = \int_{U \in G} \mathrm{d}\mu_G(U) \frac{1}{\det(id - z \cdot U)} \frac{1}{\det(id - \overline{z} \cdot U^t)}$$

$$= \alpha \oint_{\Gamma_u} \oint_{\Gamma_v} \oint_{\Gamma_w} \oint_{\Gamma_x} \frac{(1-u^2)(1-v^2)(1-w^2)(1-x^2)}{\prod\limits_{a,b,c,d \in \{1,-1\}} (1 - z \cdot u^a v^b w^c x^d)\,(1 - \overline{z} \cdot u^a v^b w^c x^d)} \frac{\mathrm{d}u}{u} \frac{\mathrm{d}v}{v} \frac{\mathrm{d}w}{w} \frac{\mathrm{d}x}{x}$$

($G = SU(2)^{\otimes 4}$, Γ =complex unit circle, $\alpha = 1/(2\pi i)^4$, $U = U_1 \otimes U_2 \otimes U_3 \otimes U_4$)

Four Qubits: F-Series of $SU(2)^{\otimes 4}$

$$F(z,\overline{z}) = \Big((z^{36}\overline{z}^{36} - z^{35}\overline{z}^{33} + 2z^{34}\overline{z}^{34} + 6z^{34}\overline{z}^{32} + 9z^{34}\overline{z}^{30} + 4z^{34}\overline{z}^{28} + 3z^{34}\overline{z}^{26} - z^{33}\overline{z}^{35} + 7z^{33}\overline{z}^{33} + 12z^{33}\overline{z}^{31} + \ldots + 12z^{3}\overline{z}^{5} + 7z^{3}\overline{z}^{3} - z^{3}\overline{z} + 3z^{2}\overline{z}^{10} + 4z^{2}\overline{z}^{8} + 9z^{2}\overline{z}^{6} + 6z^{2}\overline{z}^{4} + 2z^{2}\overline{z}^{2} - z\overline{z}^{3} + 1\Big) \Big/ \Big((1-\overline{z}^{6})(1-\overline{z}^{4})(1-\overline{z}^{4})(1-\overline{z}^{2})(1-z^{6})(1-z^{4})(1-z^{4})(1-z^{2}) (1-z^{3}\overline{z}^{3})(1-z^{2}\overline{z}^{2})^{4}(1-z\overline{z})(1-z^{5}\overline{z})(1-z^{3}\overline{z})^{3}(1-z^{4}\overline{z}^{2})(1-\overline{z}^{5}z) (1-\overline{z}^{3}z)^{3}(1-\overline{z}^{4}z^{2})\Big)$$

$$= 1+z^{2}+z\overline{z}+\overline{z}^{2}+3z^{4}+3z^{3}\overline{z}+8z^{2}\overline{z}^{2}+3z\overline{z}^{3}+3\overline{z}^{4}+4z^{6}+6z^{5}\overline{z}+19z^{4}\overline{z}^{2} +20z^{3}\overline{z}^{3}+19z^{2}\overline{z}^{4}+6z\overline{z}^{5}+4\overline{z}^{6}+7z^{8}+11z^{7}\overline{z}+47z^{6}\overline{z}^{2}+62z^{5}\overline{z}^{3}+98z^{4}\overline{z}^{4} +62z^{3}\overline{z}^{5}+47z^{2}\overline{z}^{6}+11z\overline{z}^{7}+7\overline{z}^{8}+9z^{10}+18z^{9}\overline{z}+81z^{8}\overline{z}^{2}+150z^{7}\overline{z}^{3} +278z^{6}\overline{z}^{4}+293z^{5}\overline{z}^{5}+278z^{4}\overline{z}^{6}+150z^{3}\overline{z}^{7}+81z^{2}\overline{z}^{8}+18z\overline{z}^{9}+9\overline{z}^{10}+14z^{12} +27z^{11}\overline{z}+143z^{10}\overline{z}^{2}+299z^{9}\overline{z}^{3}+669z^{8}\overline{z}^{4}+900z^{7}\overline{z}^{5}+1128z^{6}\overline{z}^{6}+900z^{5}\overline{z}^{7} +669z^{4}\overline{z}^{8}+299z^{3}\overline{z}^{9}+143z^{2}\overline{z}^{10}+27z\overline{z}^{11}+14\overline{z}^{12}+\ldots$$

Four Qubits: Molien Series of $U(2)^{\otimes 4}$

$$M(z) = \Big(z^{38}+6z^{35}+46z^{34}+110z^{33}+344z^{32}+844z^{31}+2154z^{30}+4606z^{29}+9397z^{28}$$
$$+16848z^{27}+28747z^{26}+44580z^{25}+65366z^{24}+88036z^{23}+111909z^{22}$$
$$+131368z^{21}+145676z^{20}+149860z^{19}+145676z^{18}+131368z^{17}$$
$$+111909z^{16}+88036z^{15}+65366z^{14}+44580z^{13}+28747z^{12}+16848z^{11}$$
$$+9397z^{10}+4606z^{9}+2154z^{8}+844z^{7}+344z^{6}+110z^{5}+46z^{4}+6z^{3}+1\Big) \Big/$$
$$\Big(\left(1-z^5\right)\left(1-z^4\right)^4\left(1-z^3\right)^6\left(1-z^2\right)^7(1-z)\Big)$$

$$= 1 + z + 8z^2 + 20z^3 + 98z^4 + 293z^5 + 1\,128z^6 + 3\,409z^7 + 10\,846z^8$$
$$+ 30\,480z^9 + 84\,652z^{10} + 217\,677z^{11} + 544\,312z^{12} + 1\,289\,225z^{13}$$
$$+ 2\,961\,626z^{14} + 6\,528\,284z^{15} + 13\,980\,717z^{16} + 28\,963\,980z^{17}$$
$$+ 58\,464\,510z^{18} + 114\,806\,429z^{19} + \ldots$$